

A NEW GENERALIZATION OF SOME INTEGRAL INEQUALITIES FOR (α, m) -CONVEX FUNCTIONS

İMDAT İŞCAN

ABSTRACT. In this paper, we derive new estimates for the remainder term of the midpoint, trapezoid, and Simpson formulae for functions whose derivatives in absolute value at certain power are (α, m) -convex.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality is well known in the literature as Hermite-Hadamard integral inequality

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

The class of (α, m) -convex functions was first introduced In [2], and it is defined as follows:

The function $f : [0, b] \rightarrow \mathbb{R}$, $b > 0$, is said to be (α, m) -convex where $(\alpha, m) \in [0, 1]^2$, if we have

$$(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

It can be easily that for $(\alpha, m) \in \{(0, 0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$ one obtains the following classes of functions: increasing, α -starshaped, starshaped, m -convex, convex, α -convex.

Denote by $K_m^\alpha(b)$ the set of all (α, m) -convex functions on $[0, b]$ for which $f(0) \leq 0$. For recent results and generalizations concerning (α, m) -convex functions (see [1, 2, 3, 4, 5, 10]).

In [10], Set et al. proved the following Hadamard type inequality for (α, m) -convex functions:

Theorem 1. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be an (α, m) -convex function with $(\alpha, m) \in (0, 1]^2$. If $0 \leq a < b < \infty$ and $f \in L[a, b]$, then one has the inequality:*

$$(1.2) \quad \frac{1}{b-a} \int_a^b f(x)dx \leq \min \left\{ \frac{f(a) + \alpha m f\left(\frac{b}{m}\right)}{\alpha + 1}, \frac{f(b) + \alpha m f\left(\frac{a}{m}\right)}{\alpha + 1} \right\}.$$

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The following inequality is well known in the literature as Simpson's inequality .

Let $f : [a, b] \rightarrow \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\|f^{(4)}\|_{\infty} = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:

$$\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b-a)^2.$$

In recent years many authors have studied error estimations for Simpson's inequality; for refinements, counterparts, generalizations and new Simpson's type inequalities, see [6, 7, 8, 9].

In this paper, in order to provide a unified approach to establish midpoint inequality, trapezoid inequality and Simpson's inequality for functions whose derivatives in absolute value at certain power are (α, m) -convex, we derive a general integral identity for convex functions.

2. MAIN RESULTS

In order to generalize the classical Trapezoid, midpoint and Simpson type inequalities and prove them, we need the following Lemma:

Lemma 1. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$, $\lambda, \mu \in [0, 1]$ and $m \in (0, 1]$. Then the following equality holds:*

$$\begin{aligned} (2.1) \quad & \lambda(\mu f(a) + (1-\mu)f(mb)) + (1-\lambda)f(\mu a + m(1-\mu)b) - \frac{1}{mb-a} \int_a^{mb} f(x) dx \\ &= (mb-a) \left[\int_0^{\mu} [-t + \lambda(1-\mu)] f'(ta + m(1-t)b) dt \right. \\ & \quad \left. + \int_{\mu}^1 [-t + (1-\alpha\lambda)] f'(ta + m(1-t)b) dt \right]. \end{aligned}$$

A simple proof of the equality can be done by performing an integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader.

Theorem 2. *Let $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I^{\circ}$ with $a < b$ and $\lambda, \mu \in [0, 1]$. If $|f'|^q$ is (α, m) -convex*

on $[a, b]$, for $(\alpha, m) \in (0, 1]^2$, $mb > a$, $q \geq 1$, then the following inequality holds:

$$(2.2) \left| \lambda (\mu f(a) + (1 - \mu) f(mb)) + (1 - \lambda) f(\mu a + m(1 - \mu)b) - \frac{1}{mb - a} \int_a^{mb} f(x) dx \right|$$

$$\leq \begin{cases} (mb - a) \left\{ \varepsilon_2^{1-\frac{1}{q}} (\delta_3 |f'(a)|^q + m\delta_4 |f'(b)|^q)^{\frac{1}{q}} \right. \\ \quad \left. + \varepsilon_3^{1-\frac{1}{q}} (\beta_1 |f'(a)|^q + m\beta_2 |f'(b)|^q)^{\frac{1}{q}} \right\}, & \lambda(1 - \mu) \leq \mu \leq 1 - \lambda\mu \\ (mb - a) \left\{ \varepsilon_1^{1-\frac{1}{q}} (\delta_1 |f'(a)|^q + m\delta_2 |f'(b)|^q)^{\frac{1}{q}} \right. \\ \quad \left. + \varepsilon_3^{1-\frac{1}{q}} (\beta_1 |f'(a)|^q + m\beta_2 |f'(b)|^q)^{\frac{1}{q}} \right\}, & \mu \leq \lambda(1 - \mu) \leq 1 - \lambda\mu, \\ (mb - a) \left\{ \varepsilon_2^{1-\frac{1}{q}} (\delta_3 |f'(a)|^q + m\delta_4 |f'(b)|^q)^{\frac{1}{q}} \right. \\ \quad \left. + \varepsilon_4^{1-\frac{1}{q}} (\beta_3 |f'(a)|^q + m\beta_4 |f'(b)|^q)^{\frac{1}{q}} \right\}, & \lambda(1 - \mu) \leq 1 - \lambda\mu \leq \mu \end{cases}$$

where

$$\varepsilon_1 = -\frac{\mu^2}{2} + \lambda(1 - \mu)\mu, \quad \varepsilon_2 = [\lambda(1 - \mu)]^2 - \varepsilon_1,$$

$$\varepsilon_3 = (1 - \lambda\mu)^2 - (1 - \lambda\mu)(1 + \mu) + \frac{1 + \mu^2}{2},$$

$$\varepsilon_4 = \frac{1 - \mu^2}{2} - (1 - \lambda\mu)(1 - \mu),$$

$$\delta_1 = \frac{\lambda(1 - \mu)\mu^{\alpha+1}}{\alpha + 1} - \frac{\mu^{\alpha+2}}{\alpha + 2},$$

$$\delta_2 = \lambda(1 - \mu)\mu - \frac{\mu^2}{2} - \delta_1,$$

$$\delta_3 = \frac{2[\lambda(1 - \mu)]^{\alpha+2}}{(\alpha + 1)(\alpha + 2)} - \frac{\lambda(1 - \mu)\mu^{\alpha+1}}{\alpha + 1} + \frac{\mu^{\alpha+2}}{\alpha + 2},$$

$$\delta_4 = [\lambda(1 - \mu)]^2 - \lambda\mu(1 - \mu) + \frac{\mu^2}{2} - \delta_3,$$

$$\beta_1 = \frac{2(1 - \lambda\mu)^{\alpha+2}}{(\alpha + 1)(\alpha + 2)} - \frac{(1 - \lambda\mu)(1 + \mu^{\alpha+1})}{\alpha + 1} + \frac{1 + \mu^{\alpha+2}}{\alpha + 2},$$

$$\beta_2 = (1 - \lambda\mu)^2 - (1 - \lambda\mu)(1 + \mu) + \frac{1 + \mu^2}{2} - \beta_1,$$

$$\beta_3 = \frac{1 - \mu^{\alpha+2}}{\alpha + 2} - (1 - \lambda\mu) \frac{1 - \mu^{\alpha+1}}{\alpha + 1},$$

$$\beta_4 = (1 - \lambda\mu)(\mu - 1) + \frac{1 - \mu^2}{2} - \beta_3.$$

Proof. From Lemma 1 and using the properties of modulus and the well known power mean inequality, we have

$$\begin{aligned}
& \left| \lambda (\mu f(a) + (1 - \mu) f(mb)) + (1 - \lambda) f(\mu a + m(1 - \mu)b) - \frac{1}{mb - a} \int_a^{mb} f(x) dx \right| \\
& \leq (mb - a) \left[\int_0^\mu |-t + \lambda(1 - \mu)| |f'(ta + m(1 - t)b)| dt \right. \\
& \quad \left. + \int_\mu^1 |-t + (1 - \lambda\mu)| |f'(ta + m(1 - t)b)| dt \right] \\
& \leq (mb - a) \left\{ \left(\int_0^\mu |-t + \lambda(1 - \mu)| dt \right)^{1 - \frac{1}{q}} \left(\int_0^\mu |-t + \lambda(1 - \mu)| |f'(ta + m(1 - t)b)|^q dt \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left(\int_\mu^1 |-t + (1 - \lambda\mu)| dt \right)^{1 - \frac{1}{q}} \left(\int_\mu^1 |-t + (1 - \lambda\mu)| |f'(ta + m(1 - t)b)|^q dt \right)^{\frac{1}{q}} \right\}
\end{aligned} \tag{2.3}$$

Since $|f'|^q$ is (α, m) -convex on $[a, b]$, we know that for $t \in [0, 1]$

$$|f'(ta + m(1 - t)b)|^q \leq t^\alpha |f'(a)|^q + m(1 - t^\alpha) |f'(b)|^q$$

hence, by simple computation

$$\int_0^\mu |-t + \lambda(1 - \mu)| dt = \begin{cases} \varepsilon_1, & \mu \leq \lambda(1 - \mu) \\ \varepsilon_2, & \mu \geq \lambda(1 - \mu) \end{cases}, \tag{2.4}$$

$$\int_\mu^1 |-t + (1 - \lambda\mu)| dt = \begin{cases} \varepsilon_3, & \mu \leq 1 - \lambda\mu \\ \varepsilon_4, & \mu \geq 1 - \lambda\mu \end{cases}, \tag{2.5}$$

$$\begin{aligned}
& \int_0^\mu |-t + \lambda(1 - \mu)| |f'(ta + m(1 - t)b)|^q dt \\
& \leq \int_0^\mu |-t + \lambda(1 - \mu)| [t^\alpha |f'(a)|^q + m(1 - t^\alpha) |f'(b)|^q] dt \\
& = \begin{cases} \delta_1 |f'(a)|^q + m\delta_2 |f'(b)|^q, & \mu \leq \lambda(1 - \mu) \\ \delta_3 |f'(a)|^q + m\delta_4 |f'(b)|^q, & \mu \geq \lambda(1 - \mu) \end{cases},
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
 & \int_{\mu}^1 |-t + (1 - \lambda\mu)| |f'(ta + m(1 - t)b)|^q dt \\
 & \leq \int_{\mu}^1 |-t + (1 - \lambda\mu)| [t^\alpha |f'(a)|^q + m(1 - t^\alpha) |f'(b)|^q] dt \\
 (2.7) \quad & = \begin{cases} \beta_1 |f'(a)|^q + m\beta_2 |f'(b)|^q, & \mu \leq 1 - \lambda\mu \\ \beta_3 |f'(a)|^q + m\beta_4 |f'(b)|^q, & \mu \geq 1 - \lambda\mu \end{cases}.
 \end{aligned}$$

Thus, using (2.4)-(2.7) in (2.3), we obtain the inequality (2.2). This completes the proof. \square

Corollary 1. *Under the assumptions of Theorem 2 with $q = 1$*

$$\begin{aligned}
 & \left| \lambda(\mu f(a) + (1 - \mu)f(mb)) + (1 - \lambda)f(\mu a + m(1 - \mu)b) - \frac{1}{mb - a} \int_a^{mb} f(x) dx \right| \\
 & \leq \begin{cases} (mb - a) \{(\delta_3 + \beta_1) |f'(a)| + m(\delta_4 + \beta_2) |f'(b)|\}, & \lambda(1 - \mu) \leq \mu \leq 1 - \lambda\mu \\ (mb - a) \{(\delta_1 + \beta_1) |f'(a)| + m(\delta_2 + \beta_2) |f'(b)|\}, & \mu \leq \lambda(1 - \mu) \leq 1 - \lambda\mu \\ (mb - a) \{(\delta_3 + \beta_3) |f'(a)| + m(\delta_4 + \beta_4) |f'(b)|\}, & \lambda(1 - \mu) \leq 1 - \lambda\mu \leq \mu \end{cases}.
 \end{aligned}$$

Remark 1. *In Corollary 1, (i) If we choose $\mu = \frac{1}{2}$, $\lambda = \frac{1}{3}$ and $\alpha = 1$, we have*

$$\begin{aligned}
 & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a + mb}{2}\right) + f(mb) \right] - \frac{1}{mb - a} \int_a^{mb} f(x) dx \right| \\
 & \leq \frac{5}{72} (mb - a) \{|f'(a)| + m|f'(b)|\}
 \end{aligned}$$

which is the same of the Simpson type inequality in [6, Corollary 2.3 (ii)].

(ii) If we choose $\mu = \frac{1}{2}$, $\lambda = 1$ and $\alpha = 1$, we have

$$\begin{aligned}
 & \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb - a} \int_a^{mb} f(x) dx \right| \\
 & \leq \frac{(mb - a)}{8} \{|f'(a)| + m|f'(b)|\}
 \end{aligned}$$

which is the same of the trapezoid type inequality in [6, Corollary 2.3 (i)].

(iii) If we choose $\mu = \frac{1}{2}$, $\lambda = 0$ and $\alpha = 1$, we have

$$\begin{aligned}
 & \left| f\left(\frac{a + mb}{2}\right) - \frac{1}{mb - a} \int_a^{mb} f(x) dx \right| \\
 & \leq \frac{(mb - a)}{8} \{|f'(a)| + m|f'(b)|\}.
 \end{aligned}$$

Corollary 2. *Under the assumptions of Theorem 2 with $\mu = \frac{1}{2}$ and $\lambda = \frac{1}{3}$, we have*

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+mb}{2}\right) + f(mb) \right] - \frac{1}{mb-a} \int_a^{mb} f(x)dx \right| \\ & \leq (mb-a) \left(\frac{5}{72}\right)^{1-\frac{1}{q}} \left\{ \left(\delta_3^* |f'(a)|^q + m \left(\frac{5}{72} - \delta_3^* \right) |f'(b)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\beta_1^* |f'(a)|^q + m \left(\frac{5}{72} - \beta_1^* \right) |f'(b)|^q \right)^{\frac{1}{q}} \right\}, \end{aligned}$$

where

$$\delta_3^* = \frac{2 + 3^{\alpha+1} (2\alpha + 1)}{6^{\alpha+2} (\alpha + 1) (\alpha + 2)}$$

and

$$\beta_1^* = \frac{2 \times 5^{\alpha+2} + 6^{\alpha+1} (\alpha - 4) - 3^{\alpha+1} (2\alpha + 7)}{6^{\alpha+2} (\alpha + 1) (\alpha + 2)}.$$

Remark 2. *In Corollary 2, if we take $\alpha = m = 1$, we obtain the following inequality*

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ & \leq (b-a) \left(\frac{5}{72}\right)^{1-\frac{1}{q}} \left\{ \left(\frac{29}{1296} |f'(b)|^q + \frac{61}{1296} |f'(a)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{61}{1296} |f'(b)|^q + \frac{29}{1296} |f'(a)|^q \right)^{\frac{1}{q}} \right\}, \end{aligned}$$

which is the same of the inequality in [8, Theorem 10] for $s = 1$.

Corollary 3. *Under the assumptions of Theorem 2 with $\mu = \frac{1}{2}$ and $\lambda = 1$, we have*

$$\begin{aligned} & \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x)dx \right| \leq (mb-a) \left(\frac{1}{8}\right)^{1-\frac{1}{q}} \\ & \times \left\{ \left(\frac{1}{2^{\alpha+2} (\alpha + 1) (\alpha + 2)} |f'(a)|^q + m \left(\frac{1}{8} - \frac{1}{2^{\alpha+2} (\alpha + 1) (\alpha + 2)} \right) |f'(b)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{\alpha 2^{\alpha+1} + 1}{2^{\alpha+2} (\alpha + 1) (\alpha + 2)} |f'(a)|^q + m \left(\frac{1}{8} - \frac{\alpha 2^{\alpha+1} + 1}{2^{\alpha+2} (\alpha + 1) (\alpha + 2)} \right) |f'(b)|^q \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Corollary 4. *Under the assumptions of Theorem 2 with $\mu = \frac{1}{2}$ and $\lambda = 0$, we have*

$$\begin{aligned} & \left| f\left(\frac{a+mb}{2}\right) - \frac{1}{mb-a} \int_a^{mb} f(x)dx \right| \\ & \leq (mb-a) \left(\frac{1}{8}\right)^{1-\frac{1}{q}} \times \left\{ \left(\frac{1}{2^{\alpha+2}(\alpha+2)} |f'(a)|^q + m \left(\frac{1}{8} - \frac{1}{2^{\alpha+2}(\alpha+2)}\right) |f'(b)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{2^{\alpha+2}-\alpha-3}{2^{\alpha+2}(\alpha+1)(\alpha+2)} |f'(a)|^q + m \left(\frac{1}{8} - \frac{2^{\alpha+2}-\alpha-3}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\right) |f'(b)|^q \right)^{\frac{1}{q}} \right\} \end{aligned}$$

Theorem 3. *Let $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$ and $\lambda, \mu \in [0, 1]$. If $|f'|^q$ is (α, m) -convex on $[a, b]$, for $(\alpha, m) \in (0, 1]^2$, $mb > a$, $q > 1$, then the following inequality holds:*

$$\begin{aligned} & \left| \lambda(\mu f(a) + (1-\mu)f(mb)) + (1-\lambda)f(\mu a + m(1-\mu)b) - \frac{1}{mb-a} \int_a^{mb} f(x)dx \right| \\ & \leq (mb-a) \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \cdot \begin{cases} \left[\vartheta_1^{\frac{1}{p}} A^{\frac{1}{q}} + \vartheta_3^{\frac{1}{p}} B^{\frac{1}{q}} \right], & \lambda(1-\mu) \leq \mu \leq 1-\lambda\mu \\ \left[\vartheta_2^{\frac{1}{p}} A^{\frac{1}{q}} + \vartheta_3^{\frac{1}{p}} B^{\frac{1}{q}} \right], & \mu \leq \lambda(1-\mu) \leq 1-\lambda\mu \\ \left[\vartheta_2^{\frac{1}{p}} A^{\frac{1}{q}} + \vartheta_4^{\frac{1}{p}} B^{\frac{1}{q}} \right], & \lambda(1-\mu) \leq 1-\lambda\mu \leq \mu \end{cases} \end{aligned}$$

where

$$A = \mu \times \min \left\{ \frac{|f'(\mu a + m(1-\mu)b)|^q + \alpha m |f'(b)|^q}{\alpha+1}, \frac{|f'(mb)|^q + \alpha m \left| f'\left(\frac{\mu a + m(1-\mu)b}{m}\right) \right|^q}{\alpha+1} \right\}$$

$$B = (1-\mu) \times \min \left\{ \frac{|f'(\mu a + m(1-\mu)b)|^q + \alpha m \left| f'\left(\frac{a}{m}\right) \right|^q}{\alpha+1}, \frac{|f'(a)|^q + \alpha m \left| f'\left(\frac{\mu a + m(1-\mu)b}{m}\right) \right|^q}{\alpha+1} \right\}$$

$$\begin{aligned} \vartheta_1 &= [\lambda(1-\mu)]^{p+1} + [\mu - \lambda(1-\mu)]^{p+1}, \quad \vartheta_2 = [\lambda(1-\mu)]^{p+1} - [\lambda(1-\mu) - \mu]^{p+1}, \\ \vartheta_3 &= [1 - \lambda\mu - \mu]^{p+1} + [\lambda\mu]^{p+1}, \quad \vartheta_4 = [\lambda\mu]^{p+1} - [\mu - 1 + \lambda\mu]^{p+1}, \end{aligned}$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 1 and by Hölder's integral inequality, we have

$$\begin{aligned}
& \left| \lambda (\mu f(a) + (1 - \mu) f(mb)) + (1 - \lambda) f(\mu a + m(1 - \mu) b) - \frac{1}{mb - a} \int_a^{mb} f(x) dx \right| \\
& \leq (mb - a) \left[\int_0^\mu |-t + \lambda(1 - \mu)| |f'(ta + m(1 - t)b)| dt \right. \\
& \quad \left. + \int_\mu^1 |-t + (1 - \lambda\mu)| |f'(ta + m(1 - t)b)| dt \right] \\
& \leq (mb - a) \left\{ \left(\int_0^\mu |-t + \lambda(1 - \mu)|^p dt \right)^{\frac{1}{p}} \left(\int_0^\mu |f'(ta + m(1 - t)b)|^q dt \right)^{\frac{1}{q}} \right. \\
& \quad \left. (2.9) + \left(\int_\mu^1 |-t + (1 - \lambda\mu)|^p dt \right)^{\frac{1}{p}} \left(\int_\mu^1 |f'(ta + m(1 - t)b)|^q dt \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

Since $|f'|^q$ is (α, m) -convex on $[a, b]$, for $(\alpha, m) \in (0, 1]^2$ and $\mu \in (0, 1]$ by the inequality (1.2), we get

$$\begin{aligned}
(2.10) \quad & \int_0^\mu |f'(ta + m(1 - t)b)|^q dt = \mu \left[\frac{1}{\mu(mb - a)} \int_{\mu a + m(1 - \mu)b}^{mb} |f'(x)|^q dx \right] \leq \\
& \mu \times \min \left\{ \frac{|f'(\mu a + m(1 - \mu)b)|^q + \alpha m |f'(b)|^q}{\alpha + 1}, \frac{|f'(mb)|^q + \alpha m \left| f' \left(\frac{\mu a + m(1 - \mu)b}{m} \right) \right|^q}{\alpha + 1} \right\}.
\end{aligned}$$

The inequality (2.10) holds for $\mu = 0$ too. Similarly, for $\mu \in [0, 1]$ by the inequality (1.2), we have

$$\begin{aligned}
(2.11) \quad & \int_\mu^1 |f'(ta + m(1 - t)b)|^q dt = (1 - \mu) \left[\frac{1}{(1 - \mu)(mb - a)} \int_a^{\mu a + m(1 - \mu)b} |f'(x)|^q dx \right] \leq \\
& (1 - \mu) \times \min \left\{ \frac{|f'(\mu a + m(1 - \mu)b)|^q + \alpha m \left| f' \left(\frac{a}{m} \right) \right|^q}{\alpha + 1}, \frac{|f'(a)|^q + \alpha m \left| f' \left(\frac{\mu a + m(1 - \mu)b}{m} \right) \right|^q}{\alpha + 1} \right\}.
\end{aligned}$$

The inequality (2.10) holds for $\mu = 1$ too. By simple computation

$$(2.12) \quad \int_0^\mu |-t + \lambda(1 - \mu)|^p dt = \begin{cases} \frac{[\lambda(1 - \mu)]^{p+1} + [\mu - \lambda(1 - \mu)]^{p+1}}{p+1}, & \lambda(1 - \mu) \leq \mu \\ \frac{[\lambda(1 - \mu)]^{p+1} - [\lambda(1 - \mu) - \mu]^{p+1}}{p+1}, & \lambda(1 - \mu) \geq \mu \end{cases},$$

and

$$(2.13) \quad \int_\mu^1 |-t + (1 - \lambda\mu)|^p dt = \begin{cases} \frac{[1 - \lambda\mu - \mu]^{p+1} + [\lambda\mu]^{p+1}}{p+1}, & \mu \leq 1 - \lambda\mu \\ \frac{[\lambda\mu]^{p+1} - [\mu - 1 + \lambda\mu]^{p+1}}{p+1}, & \mu \geq 1 - \lambda\mu \end{cases},$$

thus, using (2.10)-(2.13) in (2.9), we obtain the inequality (2.8). This completes the proof. \square

Corollary 5. *Under the assumptions of Theorem 3 with $\mu = \frac{1}{2}$ and $\lambda = \frac{1}{3}$, we have*

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+mb}{2}\right) + f(mb) \right] - \frac{1}{mb-a} \int_a^{mb} f(x)dx \right| \\ & \leq \left(\frac{mb-a}{12} \right) \left(\frac{2^{p+1}+1}{3(p+1)} \right)^{\frac{1}{p}} \left(A_1^{\frac{1}{q}} + B_1^{\frac{1}{q}} \right), \end{aligned}$$

where

$$A_1 = \min \left\{ \frac{|f'(\frac{a+mb}{2})|^q + \alpha m |f'(b)|^q}{\alpha + 1}, \frac{|f'(mb)|^q + \alpha m |f'(\frac{a+mb}{2m})|^q}{\alpha + 1} \right\},$$

and

$$B_1 = \min \left\{ \frac{|f'(\frac{a+mb}{2})|^q + \alpha m |f'(\frac{a}{m})|^q}{\alpha + 1}, \frac{|f'(a)|^q + \alpha m |f'(\frac{a+mb}{2m})|^q}{\alpha + 1} \right\}.$$

Remark 3. *In Corollary 5, if we take $\alpha = m = 1$, then we obtain the following inequality*

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \left(\frac{b-a}{12} \right) \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \\ & \times \left\{ \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(a)|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \right\}, \end{aligned}$$

which is the same of the inequality in [8, Corollary 3]

Corollary 6. *Under the assumptions of Theorem 3 with $\mu = \frac{1}{2}$ and $\lambda = 1$, we have*

$$\begin{aligned} & \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x)dx \right| \\ & \leq \left(\frac{mb-a}{4} \right) \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(A_1^{\frac{1}{q}} + B_1^{\frac{1}{q}} \right). \end{aligned}$$

Corollary 7. *Under the assumptions of Theorem 3 with $\mu = \frac{1}{2}$ and $\lambda = 0$, we have*

$$\begin{aligned} & \left| f\left(\frac{a+mb}{2}\right) - \frac{1}{mb-a} \int_a^{mb} f(x)dx \right| \\ & \leq \left(\frac{mb-a}{4} \right) \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(A_1^{\frac{1}{q}} + B_1^{\frac{1}{q}} \right). \end{aligned}$$

Theorem 4. Let $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$ and $\lambda, \mu \in [0, 1]$. If $|f'|^q$ is (α, m) -convex on $[a, b]$, for $(\alpha, m) \in (0, 1]^2$, $mb > a$, $q > 1$, then the following inequality holds:

$$(2.14) \left| \lambda (\mu f(a) + (1 - \mu) f(mb)) + (1 - \lambda) f(\mu a + m(1 - \mu)b) - \frac{1}{mb - a} \int_a^{mb} f(x) dx \right|$$

$$\leq (mb - a) \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \cdot \begin{cases} \left[\vartheta_1^{\frac{1}{p}} A_2^{\frac{1}{q}} + \vartheta_3^{\frac{1}{p}} B_2^{\frac{1}{q}} \right], & \lambda(1 - \mu) \leq \mu \leq 1 - \lambda\mu \\ \left[\vartheta_2^{\frac{1}{p}} A_2^{\frac{1}{q}} + \vartheta_3^{\frac{1}{p}} B_2^{\frac{1}{q}} \right], & \mu \leq \lambda(1 - \mu) \leq 1 - \lambda\mu \\ \left[\vartheta_2^{\frac{1}{p}} A_2^{\frac{1}{q}} + \vartheta_4^{\frac{1}{p}} B_2^{\frac{1}{q}} \right], & \lambda(1 - \mu) \leq 1 - \lambda\mu \leq \mu \end{cases},$$

where

$$A_2 = \frac{\mu^{\alpha+1} |f'(a)|^q + m [\mu(\alpha + 1) - \mu^{\alpha+1}] |f'(b)|^q}{\alpha + 1}$$

and

$$B_2 = \frac{(1 - \mu^{\alpha+1}) |f'(a)|^q + m [(\mu^{\alpha+1} - 1) + (1 - \mu)(\alpha + 1)] |f'(b)|^q}{\alpha + 1}.$$

Proof. From Lemma 1 and by Hölder's integral inequality, we have the inequality (2.9). Since $|f'|^q$ is (α, m) -convex on $[a, b]$, we know that for $t \in [0, 1 - \alpha]$ and $t \in [1 - \alpha, 1]$

$$|f'(ta + m(1 - t)b)|^q \leq t^\alpha |f'(a)|^q + m(1 - t^\alpha) |f'(b)|^q.$$

Hence

$$(2.15) \left| \lambda (\mu f(a) + (1 - \mu) f(mb)) + (1 - \lambda) f(\mu a + m(1 - \mu)b) - \frac{1}{mb - a} \int_a^{mb} f(x) dx \right|$$

$$\leq (mb - a) \left\{ \left(\int_0^\mu |-t + \lambda(1 - \mu)|^p dt \right)^{\frac{1}{p}} \left(\int_0^\mu [t^\alpha |f'(a)|^q + m(1 - t^\alpha) |f'(b)|^q] dt \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_\mu^1 |-t + (1 - \alpha\lambda)|^p dt \right)^{\frac{1}{p}} \left(\int_\mu^1 [t^\alpha |f'(a)|^q + m(1 - t^\alpha) |f'(b)|^q] dt \right)^{\frac{1}{q}} \right\}$$

$$\leq (mb - a) \left\{ \left(\int_0^\mu |-t + \lambda(1 - \mu)|^p dt \right)^{\frac{1}{p}} \left(\frac{\mu^{\alpha+1} |f'(a)|^q + m [\mu(\alpha + 1) - \mu^{\alpha+1}] |f'(b)|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_\mu^1 |-t + (1 - \alpha\lambda)|^p dt \right)^{\frac{1}{p}} \left(\frac{(1 - \mu^{\alpha+1}) |f'(a)|^q + m [(\mu^{\alpha+1} - 1) + (1 - \mu)(\alpha + 1)] |f'(b)|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right\}.$$

thus, using (2.12), (2.13) in (2.15), we obtain the inequality (2.14). This completes the proof. \square

Corollary 8. *Let the assumptions of Theorem 4 hold. Then for $\mu = \frac{1}{2}$ and $\lambda = \frac{1}{3}$, from the inequality (2.14) we get the following Simpson type inequality*

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+mb}{2}\right) + f(mb) \right] - \frac{1}{mb-a} \int_a^{mb} f(x) dx \right| \\ & \leq \left(\frac{mb-a}{12} \right) \left(\frac{2^{p+1}+1}{3(p+1)} \right)^{\frac{1}{p}} \left(\frac{1}{2^\alpha(\alpha+1)} \right)^{\frac{1}{q}} \left(A_3^{\frac{1}{q}} + B_3^{\frac{1}{q}} \right). \end{aligned}$$

where

$$A_3 = |f'(a)|^q + m[2^\alpha(\alpha+1) - 1] |f'(b)|^q$$

and

$$B_3 = (2^{\alpha+1} - 1) |f'(a)|^q + m[2^\alpha(\alpha+1) + 1 - 2^{\alpha+1}] |f'(b)|^q.$$

Corollary 9. *Let the assumptions of Theorem 4 hold. Then for $\mu = \frac{1}{2}$ and $\lambda = 1$, from the inequality (2.14) we get the following Simpson type inequality*

$$\begin{aligned} & \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) dx \right| \\ & \leq \left(\frac{mb-a}{4} \right) \left(\frac{1}{(p+1)} \right)^{\frac{1}{p}} \left(\frac{1}{2^\alpha(\alpha+1)} \right)^{\frac{1}{q}} \left(A_3^{\frac{1}{q}} + B_3^{\frac{1}{q}} \right). \end{aligned}$$

Corollary 10. *In Corollary 9, if we take $\alpha = 1$ we obtain the following inequality*

$$\begin{aligned} & \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) dx \right| \leq (mb-a) \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \\ & \times \left(\frac{1}{4} \right)^{1+\frac{1}{q}} \left[(|f'(a)|^q + 3m|f'(b)|^q)^{\frac{1}{q}} + (3|f'(a)|^q + m|f'(b)|^q)^{\frac{1}{q}} \right] \\ (2.16) \\ & \leq (mb-a) \left(\frac{1}{4} \right)^{1+\frac{1}{q}} \left[(|f'(a)|^q + 3m|f'(b)|^q)^{\frac{1}{q}} + (3|f'(a)|^q + m|f'(b)|^q)^{\frac{1}{q}} \right] \end{aligned}$$

where we have used the fact that $1/2 < (1/(p+1))^{1/p} < 1$. We note that the inequality (2.16) is the same of the inequality in [6, Corollary 2.7 (i)].

Corollary 11. *Under the assumptions of Theorem 4 with $\mu = \frac{1}{2}$ and $\lambda = 0$, we have*

$$\begin{aligned} & \left| f\left(\frac{a+mb}{2}\right) - \frac{1}{mb-a} \int_a^{mb} f(x) dx \right| \\ & \leq \left(\frac{mb-a}{4} \right) \left(\frac{1}{(p+1)} \right)^{\frac{1}{p}} \left(\frac{1}{2^\alpha(\alpha+1)} \right)^{\frac{1}{q}} \left(A_3^{\frac{1}{q}} + B_3^{\frac{1}{q}} \right). \end{aligned}$$

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DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES,, GİRESUN UNIVERSITY,
28100, GİRESUN, TURKEY.

E-mail address: `imdat.iscan@giresun.edu.tr`